

# Pure states on Cuntz algebras arising from geometric progressions

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## Abstract

Let  $\mathcal{O}_n$  denote the Cuntz algebra for  $n \geq 2$ . We introduce an embedding  $f$  of  $\mathcal{O}_m$  into  $\mathcal{O}_n$  arising from a geometric progression of Cuntz generators of  $\mathcal{O}_n$ . By identifying  $\mathcal{O}_m$  with  $f(\mathcal{O}_m)$ , we extend Cuntz states on  $\mathcal{O}_m$  to  $\mathcal{O}_n$ . We show (i) a necessary and sufficient condition of the uniqueness of the extension, (ii) the complete classification of all such extensions up to unitary equivalence of their GNS representations, and (iii) the decomposition formula of a mixing state into a convex hull of pure states. The complete set of invariants of all GNS representations by such pure states is given as a certain set of complex unit vectors.

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**Key words.** geometric progression state, geometric progression embedding, pure state, sub-Cuntz state, extension of state, Cuntz state, Cuntz algebra.

## 1 Introduction

The aim of this paper is to classify a certain class of pure states on Cuntz algebras in succession to the previous work [36]. For a unital  $C^*$ -algebra  $A$  and a unital  $C^*$ -subalgebra  $B$  of  $A$ , any state  $\omega$  on  $B$  has an *extension*  $\tilde{\omega}$  on  $A$ , that is,  $\tilde{\omega}$  is a state on  $A$  which satisfies  $\tilde{\omega}|_B = \omega$  ([16], 2.10.1), but

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it is not unique in general. In this paper, we completely classify extensions of a certain class of pure states on Cuntz algebras up to unitary equivalence of their Gel'fand-Naimark-Segal (=GNS) representations. In consequence, a new class of pure states on Cuntz algebras and the complete set of their invariants are given. In this section, we show our motivation, definitions and main theorems. Their proofs will be given in § 3.

## 1.1 Motivation

### 1.1.1 Classification problem of pure states on Cuntz algebras

A central problem of representation theory of groups is the understanding irreducible representations [39, 40]. For example, it means construction of irreducible representations, finding a complete set of invariants of representations, and understanding these invariants. Our purpose is to study irreducible representations of  $C^*$ -algebras according to such subjects. By GNS construction, the state theory of a  $C^*$ -algebra  $A$  can be interpreted as the (cyclic) representation theory of  $A$  almost all. Hence we mainly consider (pure) states instead of (irreducible) representations in this paper.

For Cuntz algebras which are typical examples of separable infinite simple  $C^*$ -algebras (see § 1.2), representations and states have been studied by many authors [3, 4, 8, 9, 11, 14, 15, 19, 20, 22, 24, 25, 36, 41, 42, 46]. They have various applications, for example, endomorphisms of  $\mathcal{B}(\mathcal{H})$  [6, 10, 21], iterated function systems [7], Markov measures [17, 18], wavelets [26], continued fractions [37], construction of  $R$ -matrices [35], construction of multiplicative isometries [33], invariant measures [28], and string theory [2]. But their classifications have not been finished yet.

We intend to develop the classification of pure states on Cuntz algebras by constructing a new class of states. For this purpose, we take notice of Cuntz states which are completely classified pure states with explicit numerical invariants (see § 1.2.2.) About other result of complete classification of states, see [31].

As a method to construct new states, we consider extensions of Cuntz states in this paper. A new idea is as follows: In the previous work [36], we classified extensions of Cuntz states on  $\mathcal{O}_{n^m}$  to  $\mathcal{O}_n$  for any integer  $m \geq 2$  with respect to a certain embedding of  $\mathcal{O}_{n^m}$  into  $\mathcal{O}_n$ . In this paper, we generalize a method to extend Cuntz states *but not* replace Cuntz states with general states. This is a crucial point to make a computable theory. For a

general embedding  $f$  of  $\mathcal{O}_m$  into  $\mathcal{O}_n$ , we introduce a new notion “ $f$ -sub-Cuntz state” as an extension of a Cuntz state on  $\mathcal{O}_m$  to  $\mathcal{O}_n$  (§ 1.2.2). As examples of  $f$ -sub-Cuntz state, we choose a certain class of embeddings (= geometric progression embeddings) and classify  $f$ -sub-Cuntz states associated with them (= geometric progression states) (§ 1.2.3). We will more closely explain this idea and its merits in the next subsection.

### 1.1.2 Extensions of Cuntz states arising from embeddings

For  $2 \leq n \leq \infty$ , let  $\mathcal{O}_n$  denote the Cuntz algebra. The outline of our strategy is as follows:

Step 1 Fix a unital embedding  $f$  of  $\mathcal{O}_m$  into  $\mathcal{O}_n$  and identify  $\mathcal{O}_m$  with  $f(\mathcal{O}_m)$ .

Step 2 Extend Cuntz states on  $\mathcal{O}_m$  to  $\mathcal{O}_n$  with respect to the inclusion  $\mathcal{O}_m \hookrightarrow \mathcal{O}_n$  in Step 1.

Step 3 Study such extensions:

- (a) For a given Cuntz state  $\omega$ , is an extension of  $\omega$  unique?
- (b) If it is unique, then write down its state values explicitly.
- (c) Find the condition of equivalence between two extensions. Furthermore, find the complete set of invariants of extensions.
- (d) If a parametrization of such (equivalence classes of) extensions are given, then investigate properties of the parametrization closely.

These will be explicitly explained in § 1.2.2 again. Merits of this scheme are as follows:

- (i) Cuntz states are well studied and they have a good parametrization. Hence it is expected that their generalizations also have good properties. For example, sub-Cuntz states are successful generalizations [36].
- (ii) If one succeeds at the proof of the uniqueness of extension, then the purity of the extended state holds automatically ([43], 4.1.7).
- (iii) Embeddings are available to study states as new tools. If one chooses adequate embeddings, then it is expected that states arising from them are computable and one can get the complete classification of them. Furthermore, one can easily generalize known theorems by generalizing related embeddings (see Definition 1.4 and (2.2)).

- (iv) It is considered that this method is a kind of induced representation theory in the broad sense of the term. Well-known Rieffel's induction [44] requires a conditional expectation (or its generalization) from an algebra to a subalgebra in order to define an induced representation. However, to find a conditional expectation is not easy except a few typical classes. Even if one does not know a conditional expectation, extensions of states always exist (Step 2) and (special classes of) embeddings can be easily constructed. Hence we expect that this scheme can play a role of alternatives of the induced representation theory of  $C^*$ -algebras.

This paper includes results in [27] by interpreting “representation” as “state”.

## 1.2 Geometric progression states

In this subsection, first, we will introduce a class of states arising from a general embedding of Cuntz algebras. Next, we will define geometric progression states as its special case. For  $2 \leq n \leq \infty$ , let  $\mathcal{O}_n$  denote the *Cuntz algebra* [12], that is,  $\mathcal{O}_n$  is a  $C^*$ -algebra which is universally generated by a (finite or infinite) sequence  $s_1, \dots, s_n$  satisfying  $s_i^* s_j = \delta_{ij} I$  for  $i, j = 1, \dots, n$  and

$$\sum_{i=1}^n s_i s_i^* = I \text{ when } n < \infty, \quad \sum_{i=1}^k s_i s_i^* \leq I, \quad k = 1, 2, \dots \text{ when } n = \infty \quad (1.1)$$

where  $I$  denotes the unit of  $\mathcal{O}_n$ . The Cuntz algebra  $\mathcal{O}_n$  is an infinite dimensional, noncommutative  $C^*$ -algebra with unit. Furthermore,  $\mathcal{O}_n$  is *simple*, that is, there exists no nontrivial closed two-sided ideal of  $\mathcal{O}_n$ .

### 1.2.1 Embeddings of Cuntz algebras

We review basics of general embeddings of Cuntz algebras. For two unital  $C^*$ -algebras  $A$  and  $B$ , let  $\text{Hom}(A, B)$  denote the set of all unital  $*$ -homomorphisms from  $A$  to  $B$ . If  $A$  is simple, then any  $f \in \text{Hom}(A, B)$  is injective, that is,  $f$  is an embedding of  $A$  into  $B$ . In this paper, we consider only unital embeddings. Let  $s_1, \dots, s_n$  denote Cuntz generators of  $\mathcal{O}_n$ . In general,  $f \in \text{Hom}(\mathcal{O}_n, A)$  is identified with Cuntz generators  $S_1, \dots, S_n$  in  $A$  as  $f(s_i) = S_i$  for  $i = 1, \dots, n$ . Hence we can define  $f$  by only images  $\{f(s_i)\}_{i=1}^n$ .

**Remark 1.1** ([30], Lemma 2.1) For  $2 \leq m, n < \infty$ ,  $\text{Hom}(\mathcal{O}_m, \mathcal{O}_n) \neq \emptyset$  if and only if  $m = (n-1)k+1$  for some  $k \geq 1$ . In this paper, we always assume the latter condition for  $(m, n)$  for a given inclusion  $\mathcal{O}_m \subset \mathcal{O}_n$ .

### 1.2.2 $f$ -sub-Cuntz states

In this subsection, we introduce  $f$ -sub-Cuntz states. For this purpose, we review Cuntz state on  $\mathcal{O}_n$ . For  $2 \leq n \leq \infty$ , let  $s_1, \dots, s_n$  denote Cuntz generators of  $\mathcal{O}_n$ . For any complex unit vector  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ , a state  $\omega_z$  on  $\mathcal{O}_n$  which satisfies

$$\omega_z(s_j) = \overline{z_j} \quad \text{for all } j = 1, \dots, n, \quad (1.2)$$

exists uniquely and is pure, where  $\overline{z_j}$  denotes the complex conjugate of  $z_j$ . When  $n = \infty$ , replace  $\mathbb{C}^n$  by  $\ell^2 := \{(z_j) : \sum_{j \geq 1} |z_j|^2 = 1\}$ . The state  $\omega_z$  is called the *Cuntz state* by  $z$  [6, 7, 10, 11]. GNS representations by  $\omega_z$  and  $\omega_y$  are unitarily equivalent if and only if  $z = y$  (see Appendix B in [36]). Bratteli and Jorgensen [7] introduced sub-Cuntz states as generalizations of Cuntz states (see § 2.2). Furthermore, we generalize sub-Cuntz states as follows.

Fix  $f \in \text{Hom}(\mathcal{O}_m, \mathcal{O}_n)$  and identify  $\mathcal{O}_m$  with  $f(\mathcal{O}_m) \subset \mathcal{O}_n$  for  $2 \leq m \leq \infty$ . With respect to this identification, an extension of a Cuntz state on  $\mathcal{O}_m$  to  $\mathcal{O}_n$  always exists. We call such a state as an  *$f$ -sub-Cuntz state* on  $\mathcal{O}_n$ . More concretely, for a unit vector  $z = (z_1, \dots, z_m) \in \mathbb{C}^m$ ,  $\omega$  is an  $f$ -sub-Cuntz state on  $\mathcal{O}_n$  by  $z$  if  $\omega$  is a state on  $\mathcal{O}_n$  which satisfies the following equations:

$$\omega(f(t_j)) = \overline{z_j} \quad \text{for all } j = 1, \dots, m \quad (1.3)$$

where  $t_1, \dots, t_m$  denote Cuntz generators of  $\mathcal{O}_m$ . When  $n = \infty$ , replace  $\mathbb{C}^n$  by  $\ell^2$ . The most essential properties of  $f$ -sub-Cuntz state are as follows.

**Lemma 1.2** Fix  $2 \leq m \leq \infty$  and  $f \in \text{Hom}(\mathcal{O}_m, \mathcal{O}_n)$ . Let  $V := \mathbb{C}^m$  when  $m < \infty$  and  $V := \ell^2$  when  $m = \infty$ . Define  $V_1 := \{x \in V : \|x\| = 1\}$ .

- (i) (*Existence*) For any  $z \in V_1$ , an  $f$ -sub-Cuntz state on  $\mathcal{O}_n$  by  $z$  exists. If it is unique, then it is pure.
- (ii) (*Equivalent conditions*) For  $z \in V_1$  and a state  $\omega$  on  $\mathcal{O}_n$  with the GNS representation  $(\mathcal{H}, \pi, \Omega)$ , the following are equivalent:
  - (a)  $\omega$  is an  $f$ -sub-Cuntz state by  $z$ .

- (b)  $\sum_j z_j \pi(f(t_j))\Omega = \Omega$ .  
(c)  $\pi(f(t_j)^*)\Omega = z_j\Omega$  for all  $j$ .

*Proof.* (i) The existence follows by definition. Since any Cuntz state is pure, there always exists a pure extension ([16], 2.10.1). Especially, if its extension is unique, then it is pure automatically.

(ii) Let  $T_j := f(t_j)$ .

(a) $\Rightarrow$ (b) If  $\omega$  is an  $f$ -sub-Cuntz state by  $z$ , then  $\sum_j z_j \omega(T_j) = 1$  from (1.3). This implies  $\|\sum_j z_j \pi(T_j)\Omega - \Omega\| = 0$ . Hence (b) is proved.

(b) $\Rightarrow$ (c) By operating  $\pi(T_j)^*$  at both sides in (b), (c) is obtained.

(c) $\Rightarrow$ (a)  $\omega(T_j) = \langle \Omega | \pi(T_j)\Omega \rangle = \langle \pi(T_j)^*\Omega | \Omega \rangle = \langle z_j\Omega | \Omega \rangle = \overline{z_j}$ . ■

**Remark 1.3** (i) An  $f$ -sub-Cuntz state by  $z$  is not always unique (see Theorem 1.5(i) or Theorem 2.4(ii)).

- (ii) Any Cuntz state on  $\mathcal{O}_n$  is an  $f$ -sub-Cuntz state with respect to  $f = id_{\mathcal{O}_n}$ .  
(iii) In (1.2), only special values of an  $f$ -sub-Cuntz state by  $z$  are given, but not its general values. Hence the determination of all values of a given  $f$ -sub-Cuntz state is one of fundamental problems.  
(iv) If  $n = m$  and  $f$  is bijective, then  $f$  is an automorphism of  $\mathcal{O}_n$ . Let  $\alpha$  denote the standard  $U(n)$ -action on  $\mathcal{O}_n$ . If  $f = \alpha_g$  for  $g \in U(n)$ , then a transformation of any Cuntz state by  $f$  is also a Cuntz state. In general, if  $f$  is an automorphism of  $\mathcal{O}_n$ , then an  $f$ -sub-Cuntz state by any  $z \in (\mathbb{C}^n)_1$  is unique.

We will show other properties of  $f$ -sub-Cuntz states in § 2.4.

### 1.2.3 Geometric progression states

In this subsection, we introduce a special class of  $f$ -sub-Cuntz states on  $\mathcal{O}_n$ . For  $2 \leq n < \infty$  and  $2 \leq m \leq \infty$ , let  $s_1, \dots, s_n$  and  $t_1, \dots, t_m$  denote Cuntz generators of  $\mathcal{O}_n$  and  $\mathcal{O}_m$ , respectively.

**Definition 1.4** (i) When  $m = (n - 1)k + 1$  for  $k \geq 1$ , define  $f \in \text{Hom}(\mathcal{O}_m, \mathcal{O}_n)$  by

$$\begin{cases} f(t_{(n-1)r+i}) := s_n^r s_i & \begin{pmatrix} r = 0, 1, \dots, k-1, \\ i = 1, \dots, n-1 \end{pmatrix}, \\ f(t_m) := s_n^k. \end{cases} \quad (1.4)$$

(ii) When  $m = \infty$ , define  $f \in \text{Hom}(\mathcal{O}_\infty, \mathcal{O}_n)$  by

$$f(t_{(n-1)r+i}) := s_n^r s_i \quad (r \geq 0, i = 1, \dots, n-1). \quad (1.5)$$

Here  $s_n^0$  denotes the unit of  $\mathcal{O}_n$  for convenience. We call  $f$  the geometric progression embedding of  $\mathcal{O}_m$  into  $\mathcal{O}_n$ .

For example, when  $(n, m) = (2, 3)$ ,  $f$  in (1.4) is given as follows:

$$f(t_1) = s_1, \quad f(t_2) = s_2 s_1, \quad f(t_3) = s_2^2. \quad (1.6)$$

When  $n = 2$  in (1.5),  $f(t_i)$ 's are given as follows:

$$s_1, s_2 s_1, s_2^2 s_1, s_2^3 s_1, \dots, s_2^r s_1, \dots \quad (1.7)$$

This is the origin of “geometric progression embedding”. Geometric progression embeddings have appeared in [12, 13, 27, 29, 30, 32, 34, 38]. We expressly provide the definition of  $f$ -sub-Cuntz state for a given geometric progression embedding  $f$  according to (1.3) as follows.

(i) For  $f$  in (1.4),  $\omega$  is an  $f$ -sub-Cuntz state by  $z = (z_1, \dots, z_m) \in (\mathbb{C}^m)_1 := \{y \in \mathbb{C}^m : \|y\| = 1\}$  if and only if  $\omega$  is a state on  $\mathcal{O}_n$  which satisfies

$$\begin{cases} \omega(s_n^r s_i) = \overline{z_{(n-1)r+i}} & \text{for all } r = 0, 1, \dots, k-1 \text{ and } i = 1, \dots, n-1, \\ \omega(s_n^k) = \overline{z_m}. \end{cases} \quad (1.8)$$

(ii) For  $f$  in (1.5),  $\omega$  is an  $f$ -sub-Cuntz state on  $\mathcal{O}_n$  for  $z = (z_1, z_2, \dots) \in \ell_1^2 := \{y \in \ell^2 : \|y\| = 1\}$  if and only if  $\omega$  is a state on  $\mathcal{O}_n$  which satisfies

$$\omega(s_n^r s_i) = \overline{z_{(n-1)r+i}} \quad \text{for all } r \geq 0, i = 1, \dots, n-1. \quad (1.9)$$

We call these *geometric progression states on  $\mathcal{O}_n$  by  $z$  of order  $k$  and of order  $\infty$*  in (1.8) and (1.9), respectively. For  $f$  in (1.4), when  $k = 1$ ,  $f = \text{id}_{\mathcal{O}_n}$ . In this case, any  $f$ -sub-Cuntz state is just a Cuntz state.

About varieties of geometric progression embeddings and states arising from them, see § 4.3.

### 1.3 Main theorems

In this subsection, we show our main theorems. From Lemma 1.2(i), remaining problems on  $f$ -sub-Cuntz states are the uniqueness, decomposition formulas of mixture states, and their equivalence. We consider these problems for states introduced in § 1.2.3.

#### 1.3.1 Uniqueness, purity and decomposition of mixture

**Theorem 1.5** (*Uniqueness*)

- (i) *Let  $m = (n - 1)k + 1$  for  $k \geq 2$ . For  $z = (z_1, \dots, z_m) \in (\mathbb{C}^m)_1$ , a geometric progression state  $\omega$  on  $\mathcal{O}_n$  by  $z$  is unique if and only if  $|z_m| < 1$ . In this case,  $\omega$  is pure. We write  $\omega$  as  $\omega'_z$  when  $|z_m| < 1$ .*
- (ii) *For any  $z \in \ell_1^2$ , a geometric progression state on  $\mathcal{O}_n$  by  $z$  is unique and pure. We write such a state as  $\omega'_z$ .*

Remark that if  $k = 1$  in Theorem 1.5(i), then  $m = n$  and  $\omega$  is just the Cuntz state by  $z$ , which is unique for all  $z \in (\mathbb{C}^n)_1$ .

**Theorem 1.6** (*Decomposition of mixture*) *For  $m = (n - 1)k + 1$  with  $k \geq 2$  and  $z = (z_1, \dots, z_m) \in (\mathbb{C}^m)_1$ , if  $|z_m| = 1$ , then a geometric progression state on  $\mathcal{O}_n$  by  $z$  is a convex hull of Cuntz states by  $(0, \dots, 0, e^{2\pi j\sqrt{-1}/k}q) \in \mathbb{C}^n$  for  $j = 1, \dots, k$  where  $q$  is a  $k$ -th root of  $z_m$ .*

Since any Cuntz state is a geometric progression state, the set of geometric progression states is closed with respect to the pure state decomposition from Theorem 1.6.

From Theorem 1.5(i) and Theorem 1.6, the following holds.

**Corollary 1.7** *For  $m = (n - 1)k + 1$  for  $k \geq 2$ , a geometric progression state  $\omega$  by  $z = (z_1, \dots, z_m) \in (\mathbb{C}^m)_1$  is pure if and only if*

- (i)  $|z_m| < 1$  or,
- (ii)  $|z_m| = 1$  and  $\omega$  is the Cuntz state by  $(0, \dots, 0, q) \in \mathbb{C}^n$  for some  $k$ -th root  $q$  of  $z_m$ .



### 1.3.2 Equivalence

For two states  $\omega$  and  $\omega'$  on  $\mathcal{O}_n$ , we write  $\omega \sim \omega'$  if their GNS representations are unitarily equivalent.

**Theorem 1.8** *Let  $\omega'_z$  be as in Theorem 1.5.*

- (i) *For  $m = (n-1)k + 1$  with  $k \geq 2$ , define  $\mathcal{W}_m := \{(w_1, \dots, w_m) \in (\mathbb{C}^m)_1 : |w_m| < 1\}$ . For  $z, y \in \mathcal{W}_m$ ,  $\omega'_z \sim \omega'_y$  if and only if  $z = y$ .*
- (ii) *For  $z, y \in \ell_1^2$ ,  $\omega'_z \sim \omega'_y$  if and only if  $z = y$ .*

From Theorem 1.8, it is shown that  $\mathcal{W}_m$  (resp.  $\ell_1^2$ ) is the complete set of invariants of unitary equivalence classes of pure geometric progression states on  $\mathcal{O}_n$  of order  $k$  (resp. of order  $\infty$ ).

Next, we show the equivalence condition between  $\omega'_z$  ( $z \in \ell_1^2$ ) and  $\omega'_y$  ( $y \in \mathcal{W}_m$ ).

**Theorem 1.9** *Let  $\{e_i\}$  denote the standard basis of  $\ell^2$  and let  $z \in \ell_1^2$ .*

- (i) *For  $y = (y_1, \dots, y_n) \in (\mathbb{C}^n)_1$ , let  $\omega_y$  be as in (1.2). Then  $\omega'_z \sim \omega_y$  if and only if  $|y_n| < 1$  and  $z = \tilde{y}$  where  $\tilde{y} \in \ell_1^2$  is defined as*

$$\tilde{y} := \sum_{r \geq 0} \sum_{i=1}^{n-1} y_n^r y_i e_{(n-1)r+i}. \quad (1.10)$$

*In this case,  $\omega'_z = \omega_y$ .*

- (ii) *Assume  $m = (n-1)k + 1$  and  $k \geq 2$ . For  $y = (y_1, \dots, y_m) \in \mathcal{W}_m$ ,  $\omega'_z \sim \omega'_y$  if and only if  $z = \tilde{y}$  where  $\tilde{y} \in \ell_1^2$  is defined as*

$$\tilde{y} := \sum_{r \geq 0} \sum_{i=1}^{m-1} y_m^r y_i e_{(m-1)r+i}. \quad (1.11)$$

*In this case,  $\omega'_z = \omega'_y$ .*

From Theorem 1.9(i) and (ii), and Theorem 1.8(ii), the complete set of invariants of all pure geometric progression states on  $\mathcal{O}_n$  is given as follows:

$$\ell_1^2 \cup \{(0, \dots, 0, c) \in \mathbb{C}^n : |c| = 1\}. \quad (1.12)$$

In other words, every pure geometric progression state is parametrized by a vector in (1.12), and for any two distinct vectors in (1.12), associated geometric progression states are not equivalent.

Next, we show relations between geometric progression states of different finite orders.

**Theorem 1.10** (i) Assume  $m = (n-1)a + 1$  for  $a \geq 1$ . Let  $z \in \mathcal{W}_m$  and  $l = (m-1)k + 1$  for some  $k \geq 1$ . Define  $\hat{z} \in \mathcal{W}_l$  by

$$\hat{z} := \sum_{r=0}^{k-1} \sum_{i=1}^{m-1} z_m^r z_i e_{(m-1)r+i} + z_m^k e_l \quad (1.13)$$

where  $\{e_i\}$  denotes the standard basis of  $\mathbb{C}^l$ . Then  $\omega'_z = \omega'_{\hat{z}}$ .

(ii) Let  $\mathcal{S}_{m,n}$  denote the set of all geometric progression states on  $\mathcal{O}_n$  parametrized by  $z \in \mathcal{W}_m$ , that is,  $\mathcal{S}_{m,n} := \{\omega'_z : z \in \mathcal{W}_m\}$ . If  $l \geq m \geq 2$  satisfy that  $m-1$  is a divisor of  $l-1$ , then  $\mathcal{S}_{m,n} \subset \mathcal{S}_{l,n}$ .

(iii) For any  $m, l \in \{(n-1)k + 1 : k \geq 2\}$ , let  $p := (m-1)(l-1) + 1$  and  $z \in \mathcal{W}_m$  and  $y \in \mathcal{W}_l$ . Then  $\omega'_z \sim \omega'_y$  if and only if the following equation of vectors in  $\mathbb{C}^p$  holds:

$$\sum_{r=0}^{l-2} \sum_{i=1}^{m-1} z_m^r z_i e_{(m-1)r+i} + z_m^{l-1} e_p = \sum_{r'=0}^{m-2} \sum_{i'=1}^{l-1} y_l^{r'} y_{i'} e_{(l-1)r'+i'} + y_l^{m-1} e_p \quad (1.14)$$

where  $\{e_i\}$  denotes the standard basis of  $\mathbb{C}^p$ .

(iv) Assume  $m = (n-1)k + 1$  and  $k \geq 2$ . Let  $z \in \mathcal{W}_m$  and  $y = (y_1, \dots, y_n) \in (\mathbb{C}^n)_1$ . Let  $\omega_y$  be as in (1.2). Then  $\omega'_z \sim \omega_y$  if and only if  $|y_n| < 1$  and  $z = \hat{y}$  where  $\hat{y} \in (\mathbb{C}^m)_1$  is defined as

$$\hat{y} := \sum_{r=0}^{k-1} \sum_{j=1}^{n-1} y_n^r y_j e_{(n-1)r+j} + y_n^k e_m \quad (1.15)$$

where  $\{e_j\}$  denotes the standard basis of  $\mathbb{C}^m$ .

### 1.3.3 Finite correlation

For  $2 \leq n < \infty$ , define

$$\mathcal{I}_n := \bigcup_{l \geq 0} \{1, \dots, n\}^l, \quad \{1, \dots, n\}^0 := \{\emptyset\}. \quad (1.16)$$

For  $n = \infty$ , replace  $\{1, \dots, n\}$  by  $\mathbb{N} := \{1, 2, \dots\}$  in (1.16). For  $J = (j_1, \dots, j_r) \in \mathcal{I}_n$ , we write  $s_J := s_{j_1} \cdots s_{j_r}$  and let  $s_\emptyset := I$  for convenience. For a state  $\omega$  on  $\mathcal{O}_n$ ,  $\omega$  is said to be *finitely correlated* [10] if the dimension of  $\mathcal{K}(\omega) := \text{Lin}\langle \{\pi(s_J)^* \Omega : J \in \mathcal{I}_n\} \rangle$  is finite where  $(\mathcal{H}, \pi, \Omega)$  denotes the GNS representation by  $\omega$ .

**Theorem 1.11** *For  $n < \infty$ , any geometric progression state  $\omega$  on  $\mathcal{O}_n$  of order  $k < \infty$  satisfies  $\dim \mathcal{K}(\omega) \leq k$ . Especially,  $\omega$  is finitely correlated.*

It is known that any sub-Cuntz state on  $\mathcal{O}_n$  ( $n < \infty$ ) is finitely correlated (see Lemma 2.5(i)). Furthermore, it is easily shown that  $\omega$  is a Cuntz state if and only if  $\dim \mathcal{K}(\omega) = 1$ .

## 1.4 Properties of state parametrization

In this subsection, we show properties of the state parametrization

$$\mathcal{W}_m \ni z \longmapsto \omega'_z \in \mathcal{P}(\mathcal{O}_n) \quad (1.17)$$

where  $\mathcal{P}(\mathcal{O}_n)$  denotes the set of all pure states on  $\mathcal{O}_n$  and  $\mathcal{W}_\infty := \ell_1^2$  when  $m = \infty$  for convenience. We consider the naturality and relevance of the parametrization (1.17). From Theorem 1.5, (1.17) is injective and it can be defined into the set of all unitary equivalence classes of pure states (or irreducible representations) of  $\mathcal{O}_n$  (= the spectrum of  $\mathcal{O}_n$  [16]) from Theorem 1.8. In addition, we show the following two properties: (i) (1.17) is covariant with respect to certain  $U(n-1)$ -actions. (ii) (1.17) is an isomorphism of two inductive systems.

### 1.4.1 $U(n-1)$ -covariance of state parametrization

We introduce two actions of unitary groups on  $\mathcal{W}_m$  and  $\mathcal{P}(\mathcal{O}_n)$  as follows. For  $2 \leq m < \infty$ , we write the standard action of  $U(m)$  on the vector space

$\mathbb{C}^m$  as  $gz$  for  $g \in U(m)$  and  $z \in \mathbb{C}^m$ . Identify  $U(m-1)$  with a subgroup of  $U(m)$  with respect to the embedding

$$U(m-1) \ni g \mapsto \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \in U(m). \quad (1.18)$$

The subgroup  $U(m-1)$  of  $U(m)$  also acts on  $\mathbb{C}^m$ . From (1.18) and the definition of  $\mathcal{W}_m$ , the subset  $\mathcal{W}_m \subset \mathbb{C}^m$  is invariant under the action of  $U(m-1)$ , that is,  $g\mathcal{W}_m \subset \mathcal{W}_m$  for all  $g \in U(m-1)$ .

On the other hand, let  $\alpha$  denote the standard  $U(n)$ -action on  $\mathcal{O}_n$  defined by  $\alpha_g(s_i) := \sum_{j=1}^n g_{ji}s_j$  for  $i = 1, \dots, n$  and  $g = (g_{ij}) \in U(n)$ . For a state  $\omega$  on  $\mathcal{O}_n$ , define  $\alpha_g^*(\omega) := \omega \circ \alpha_{g^{-1}}$  for  $g \in U(n)$ . Identify  $U(n-1)$  with a subgroup of  $U(n)$  with respect to the embedding in (1.18) by replacing  $m$  with  $n$ . By this identification,  $U(n-1)$  also acts on  $\mathcal{O}_n$  such that  $\alpha_g(s_n) = s_n$  for any  $g \in U(n-1)$ .

Then we have the following result.

**Theorem 1.12** ( $U(n-1)$ -covariance)

(i) For  $m = (n-1)k + 1$  ( $k \geq 2$ ) and  $z \in \mathcal{W}_m$ , the following holds:

$$\alpha_g^*(\omega'_z) = \omega'_{\tilde{g}z} \quad (g \in U(n-1)) \quad (1.19)$$

where  $\tilde{g} = (\tilde{g}_{i,j}) \in U(m-1) \subset U(m)$  is defined as  $\tilde{g}_{(n-1)a+i, (n-1)b+j} := \delta_{ab} g_{ij}$  for  $i, j = 1, \dots, n-1$  and  $0 \leq a, b \leq k-1$  and  $\tilde{g}_{m,m} := 1$ .

(ii) For any  $z \in \ell_1^2$ , the following holds:

$$\alpha_g^*(\omega'_z) = \omega'_{\tilde{g}z} \quad (g \in U(n-1)) \quad (1.20)$$

where  $\tilde{g} = (\tilde{g}_{i,j})$  is the unitary operator on  $\ell^2$  defined as  $\tilde{g}_{(n-1)a+i, (n-1)b+j} := \delta_{ab} g_{ij}$  for  $i, j = 1, \dots, n-1$  and  $a, b \geq 0$  where  $\tilde{g}_{i,j}$ 's denote matrix components of  $\tilde{g}$  with respect to the standard basis of  $\ell^2$ .

*Proof.* (i) By definition,  $\tilde{g}z \in \mathcal{W}_m$ . Hence  $\omega'_{\tilde{g}z}$  is uniquely defined from Theorem 1.5. Identify  $\mathcal{O}_m$  with  $f(\mathcal{O}_m)$  for  $f$  in (1.4). Then we can verify  $\{\alpha_g^*(\omega'_z)\}(t_{(n-1)r+i}) = \overline{(\tilde{g}z)_{(n-1)r+i}}$  for all  $r = 0, 1, \dots, k-1$  and  $i = 1, \dots, n-1$ , and  $\{\alpha_g^*(\omega'_z)\}(t_m) = \overline{(\tilde{g}z)_m}$ . Hence  $\alpha_g^*(\omega'_z) = \omega'_{\tilde{g}z}$  by the uniqueness of  $\omega'_{\tilde{g}z}$ .  
(ii) As the same token with the proof of (i), the statement can be proved. ■

From Theorem 1.12(i) (*resp.* (ii)), we see that the parametrization (1.17) is covariant with respect to two actions of  $U(n-1)$  on  $\mathcal{W}_m$  (*resp.*  $\ell_1^2$ ) and  $\mathcal{P}(\mathcal{O}_n)$ . When  $m = n$  in Theorem 1.12,  $\omega'_z$  is just the Cuntz state  $\omega_z$ . In this case, it is known that  $\alpha_g^*(\omega_z) = \omega_{gz}$  for all  $g \in U(n)$  and  $z \in (\mathbb{C}^n)_1$  ([36], (1.14)). Hence Theorem 1.12 is a natural generalization of this covariance.

#### 1.4.2 State parametrization as an isomorphism between two inductive systems

For a directed set  $(D, \leq)$ , a data  $\{(A_d, \varphi_{e,d}) : d, e \in D\}$  is a (set-theoretical) *inductive system* (or *directed system*) over  $(D, \leq)$  [5] if maps  $\varphi_{e,d} : A_d \rightarrow A_e$  ( $d \leq e$ ) satisfy  $\varphi_{g,e} \circ \varphi_{e,d} = \varphi_{g,d}$  ( $d \leq e \leq g$ ) and  $\varphi_{d,d} = id_{A_d}$ . For  $l, k \in \mathbb{N} := \{1, 2, \dots\}$ , if  $k$  is a divisor of  $l$ , then we write  $k \prec l$ . We introduce two inductive systems over the directed set  $(\mathbb{N}, \prec)$ .

**Theorem 1.13** *Fix  $2 \leq n < \infty$ . For  $\mathcal{S}_{m,n}$  in Theorem 1.10(ii) and  $\mathcal{W}_m$  in Theorem 1.8(i), define*

$$\mathcal{S}(k) := \mathcal{S}_{(n-1)k+1,n}, \quad \mathcal{W}(k) := \mathcal{W}_{(n-1)k+1} \quad (k \geq 1). \quad (1.21)$$

(i)  $\{(\mathcal{S}(k), \subset) : k \in \mathbb{N}\}$  is an inductive system over  $(\mathbb{N}, \prec)$  with respect to inclusions  $\subset$  in Theorem 1.10(ii).

(ii) When  $l \succ k$ , define the map  $\psi_{l,k} : \mathcal{W}(k) \rightarrow \mathcal{W}(l)$  by

$$\psi_{l,k}(z) := \hat{z} \quad (1.22)$$

where  $\hat{z}$  is as in Theorem 1.10(i). Then  $\{(\mathcal{W}(k), \psi_{l,k}) : k, l \in \mathbb{N}\}$  is an inductive system over  $(\mathbb{N}, \prec)$ .

(iii) The state parametrization

$$\Phi_k : \mathcal{W}(k) \ni z \longmapsto \omega'_z \in \mathcal{S}(k) \quad (k \geq 1) \quad (1.23)$$

gives an isomorphism  $\{\Phi_k : k \geq 1\}$  between two inductive systems in (i) and (ii).

*Proof.* (i) Assume  $k \prec l$ . From Theorem 1.10(ii),  $\mathcal{S}(k) \subset \mathcal{S}(l)$ . We see that its inclusion map  $\phi_{l,k} : \mathcal{S}(k) \hookrightarrow \mathcal{S}(l)$  satisfies  $\phi_{m,l} \circ \phi_{l,k} = \phi_{m,k}$  when  $m \succ l \succ k$ .

- (ii) By definition, we can prove  $\psi_{m,l} \circ \psi_{l,k} = \psi_{m,k}$  when  $m \succ l \succ k$ . Hence the statement holds.
- (iii) By definition,  $\Phi_k$  is a bijection. From proofs of (i) and (ii), the statement holds. ■

The paper is organized as follows. In § 2, we will review known results and prepare lemmas to prove main theorems. In § 3, we will prove main theorems. In § 4, we will show examples.

## 2 Preparations

### 2.1 Equivalent conditions of geometric progression state

For convenience, we show equivalent conditions of geometric progression state here. Let  $s_1, \dots, s_n$  denote Cuntz generators of  $\mathcal{O}_n$ . From Lemma 1.2(ii) and Definition 1.4, the following holds.

**Corollary 2.1** *Let  $\omega$  be a state on  $\mathcal{O}_n$  with the GNS representation  $(\mathcal{H}, \pi, \Omega)$ .*

- (i) *Assume  $m = (n-1)k + 1$  for  $k \geq 1$ . For  $z \in (\mathbb{C}^m)_1$ , the following are equivalent:*

- (a)  *$\omega$  is a geometric progression state by  $z$ .*
- (b) 
$$\left\{ \sum_{r=0}^{k-1} \sum_{i=1}^{n-1} z_{(n-1)r+i} \pi(s_n^r s_i) + z_m \pi(s_n^k) \right\} \Omega = \Omega.$$
- (c) 
$$\pi(s_n^r s_i)^* \Omega = z_{(n-1)r+i} \Omega \text{ for } r = 0, 1, \dots, k-1 \text{ and } i = 1, \dots, n-1,$$
  
and 
$$\pi(s_n^k)^* \Omega = z_m \Omega.$$

- (ii) *For  $z \in \ell_1^2$ , the following are equivalent:*

- (a)  *$\omega$  is a geometric progression state by  $z$ .*
- (b) 
$$\sum_{r \geq 0} \sum_{i=1}^{n-1} z_{(n-1)r+i} \pi(s_n^r s_i) \Omega = \Omega.$$
- (c) 
$$\pi(s_n^r s_i)^* \Omega = z_{(n-1)r+i} \Omega \text{ for } r \geq 0 \text{ and } i = 1, \dots, n-1.$$

For the case of  $k = 1$  in Corollary 2.1(i), we obtain the equivalent conditions of Cuntz state, that is, the following are equivalent:

- (i)  $\omega$  is the Cuntz state by  $z$ .
- (ii)  $\{z_1\pi(s_1) + \cdots + z_n\pi(s_n)\}\Omega = \Omega$ .
- (iii)  $\pi(s_i)^*\Omega = z_i\Omega$  for  $i = 1, \dots, n$ .

We show relations between Cuntz generators of  $\mathcal{O}_m$  and  $\mathcal{O}_n$ .

**Lemma 2.2** *Let  $\mathcal{I}_n$  be as in (1.16).*

- (i) *Assume  $m = (n-1)k + 1$  for  $k \geq 2$ . For  $f$  in (1.4), we write  $f(t_i)$  as  $t_i$  for short. For any  $J \in \mathcal{I}_n$ , there exists a unique pair  $(\hat{J}, a) \in \mathcal{I}_m \times \{0, 1, \dots, k-1\}$  such that  $s_J = t_{\hat{J}}s_n^a$ .*
- (ii) *For  $f$  in (1.5), we write  $f(t_i)$  as  $t_i$  for short. For any  $J \in \mathcal{I}_n$ , there exists a unique pair  $(\hat{J}, a) \in \mathcal{I}_\infty \times \mathbb{Z}_{\geq 0}$  such that  $s_J = t_{\hat{J}}s_n^a$ .*

*Proof.* See Appendix A. ■

## 2.2 Sub-Cuntz states

In this subsection, we review sub-Cuntz state [36]. For  $m \geq 1$ , let  $\mathcal{V}_{n,m}$  denote the complex Hilbert space with the orthonormal basis  $\{e_J : J \in \{1, \dots, n\}^m\}$ , that is,  $\mathcal{V}_{n,m} = \ell^2(\{1, \dots, n\}^m) \cong \mathbb{C}^{n^m}$ . Let  $(\mathcal{V}_{n,m})_1 := \{z \in \mathcal{V}_{n,m} : \|z\| = 1\}$ .

**Definition 2.3** *For  $z = \sum z_J e_J \in (\mathcal{V}_{n,m})_1$ ,  $\omega$  is a sub-Cuntz state on  $\mathcal{O}_n$  by  $z$  if  $\omega$  is a state on  $\mathcal{O}_n$  which satisfies the following equations:*

$$\omega(s_J) = \overline{z_J} \quad \text{for all } J \in \{1, \dots, n\}^m \quad (2.1)$$

where  $s_J := s_{j_1} \cdots s_{j_m}$  when  $J = (j_1, \dots, j_m)$ , and  $\overline{z_J}$  denotes the complex conjugate of  $z_J$ . In this case,  $\omega$  is called a sub-Cuntz state of order  $m$ .

A sub-Cuntz state  $\omega$  of order 1 is just a Cuntz state. A sub-Cuntz state  $\omega$  of order  $m$  is an  $f$ -sub-Cuntz state with respect to the embedding  $f \in \text{Hom}(\mathcal{O}_{n^m}, \mathcal{O}_n)$  defined by

$$f(t_i) := s_{J(i)} \quad (i = 1, \dots, n^m) \quad (2.2)$$

where  $J(i) = (j_1, \dots, j_m) \in \{1, \dots, n\}^m$  is defined as  $i = \sum_{r=1}^m (j_r - 1)n^{m-r} + 1$ . For example, if  $(n, m) = (2, 2)$ , then we obtain  $(J(1), J(2), J(3), J(4)) = ((1, 1), (1, 2), (2, 1), (2, 2))$ .

We identify  $\mathcal{V}_{n,m}$  with  $(\mathcal{V}_{n,1})^{\otimes m}$  by the correspondence between bases  $e_J \mapsto e_{j_1} \otimes \dots \otimes e_{j_m}$  for  $J = (j_1, \dots, j_m) \in \{1, \dots, n\}^m$ . From this identification, we obtain  $\mathcal{V}_{n,m} \otimes \mathcal{V}_{n,l} = \mathcal{V}_{n,m+l}$  for any  $m, l \geq 1$ . Then the following hold.

- Theorem 2.4** (i) ([36], Fact 1.3) For any  $z \in (\mathcal{V}_{n,m})_1$ , a sub-Cuntz state on  $\mathcal{O}_n$  by  $z$  exists.
- (ii) ([36], Theorem 1.4) For a sub-Cuntz state  $\omega$  on  $\mathcal{O}_n$  by  $z \in (\mathcal{V}_{n,m})_1$ ,  $\omega$  is unique if and only if  $z$  is nonperiodic, that is,  $z = x^{\otimes p}$  for some  $x$  implies  $p = 1$ . In this case,  $\omega$  is pure and we write it as  $\tilde{\omega}_z$ .
- (iii) ([36], Theorem 1.5) Let  $p \geq 2$  and  $z = x^{\otimes p}$  for a nonperiodic element  $x \in (\mathcal{V}_{n,m'})_1$ . If  $\omega$  is a sub-Cuntz state on  $\mathcal{O}_n$  by  $z$ , then  $\omega$  is a convex hull of sub-Cuntz states by  $e^{2\pi j \sqrt{-1}/p} x$  for  $j = 1, \dots, p$ .
- (iv) ([36], Theorem 1.7) For  $z, y \in \bigcup_{m \geq 1} (\mathcal{V}_{n,m})_1$ , assume that both  $z$  and  $y$  are nonperiodic. Then the following are equivalent:
- (a) GNS representations by  $\tilde{\omega}_z$  and  $\tilde{\omega}_y$  are unitarily equivalent.
  - (b)  $z$  and  $y$  are conjugate, that is,  $z = y$ , or  $z = x_1 \otimes x_2$  and  $y = x_2 \otimes x_1$  for some  $x_1, x_2 \in \bigcup_{m \geq 1} (\mathcal{V}_{n,m})_1$ .

About concrete examples, see Example 4.3 (see also § 4 in [36]).

- Lemma 2.5** (i) ([36], Lemma 2.4(i)) When  $n < \infty$ , any sub-Cuntz state on  $\mathcal{O}_n$  is finitely correlated.
- (ii) ([36], Lemma 2.4(ii)) If  $\omega$  is a sub-Cuntz state with the GNS representation  $(\mathcal{H}, \pi, \Omega)$ , then  $\text{Lin}\langle \{\pi(s_J)\Omega : J \in \mathcal{I}_n\} \rangle$  is dense in  $\mathcal{H}$ .
- (iii) ([36], Theorem 2.3) Fix  $m \geq 1$ . Let  $\omega$  be a state on  $\mathcal{O}_n$  with the GNS representation  $(\mathcal{H}, \pi, \Omega)$ . For  $z = \sum z_J e_J \in (\mathcal{V}_{n,m})_1$ , the following are equivalent:
- (a)  $\omega$  is a sub-Cuntz state by  $z$ .
  - (b)  $\Omega = \pi(s(z))\Omega$  where  $s(z) := \sum z_J s_J$ .
  - (c)  $\pi(s_J)^* \Omega = z_J \Omega$  for all  $J \in \{1, \dots, n\}^m$ .



## 2.3 GNS representations by geometric progression states

In this subsection, we show properties of GNS representations by geometric progression states. Let  $\mathcal{I}_n$  be as in (1.16). For  $J = (j_1, \dots, j_r) \in \mathcal{I}_n$  and  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ , we write  $z_J := z_{j_1} \cdots z_{j_r}$  and  $z_\emptyset := 1$  for convenience.

**Lemma 2.6** *Assume  $m = (n-1)k + 1$  for  $k \geq 2$ . For  $z \in (\mathbb{C}^m)_1$ , let  $\omega$  be a geometric progression state on  $\mathcal{O}_n$  by  $z$  with the GNS representation  $(\mathcal{H}, \pi, \Omega)$ .*

- (i) *For any  $J \in \mathcal{I}_n$ ,  $\pi(s_J)^*\Omega \in \text{Lin}\langle\{\pi(s_n^a)^*\Omega : a = 0, \dots, k-1\}\rangle$ .*
- (ii) *For any  $J \in \mathcal{I}_n$ ,  $\pi(s_J)^*\Omega \in \text{Lin}\langle\{\pi(s_L)\Omega : L \in \mathcal{I}_n\}\rangle$ .*
- (iii)  *$\text{Lin}\langle\{\pi(s_L)\Omega : L \in \mathcal{I}_n\}\rangle$  is dense in  $\mathcal{H}$ .*

*Proof.* (i) From Lemma 2.2(i) and Lemma 1.2(ii)((a) $\Rightarrow$ (c)), we obtain

$$\pi(s_J)^*\Omega = \pi(t_J s_n^a)^*\Omega = \pi(s_n^a)^*\pi(t_J)^*\Omega = \overline{z_J} \pi(s_n^a)^*\Omega \quad (2.3)$$

for some  $(\hat{J}, a)$  with  $0 \leq a \leq k-1$ . Hence the statement holds.

(ii) From Corollary 2.1(i)((a) $\Rightarrow$ (b)), we can prove

$$\pi(s_n^a)^*\Omega = \left\{ \sum_{r=a}^{k-1} \sum_{i=1}^{n-1} z_{(n-1)r+i} \pi(s_n^{r-a} s_i) + z_m \pi(s_n^{k-a}) \right\} \Omega. \quad (2.4)$$

From this and (i), the statement holds.

(iii) Since  $\text{Lin}\langle\{\pi(s_J s_K^*)\Omega : J, K \in \mathcal{I}_n\}\rangle$  is dense in  $\mathcal{H}$ , the statement holds from (ii). ■

**Lemma 2.7** *For  $z \in \ell_1^2$ , let  $\omega$  be a geometric progression state on  $\mathcal{O}_n$  by  $z$  with the GNS representation  $(\mathcal{H}, \pi, \Omega)$ .*

- (i) *For any  $J \in \mathcal{I}_n$ ,  $\pi(s_J)^*\Omega \in \text{Lin}\langle\{\pi(s_n^a)^*\Omega : a \geq 0\}\rangle$ .*
- (ii) *For any  $J \in \mathcal{I}_n$ ,  $\pi(s_J)^*\Omega \in \overline{\text{Lin}\langle\{\pi(s_L)\Omega : L \in \mathcal{I}_n\}\rangle}$ .*
- (iii)  *$\text{Lin}\langle\{\pi(s_L)\Omega : L \in \mathcal{I}_n\}\rangle$  is dense in  $\mathcal{H}$ .*

*Proof.* By replacing “Lemma 2.2(i)” in the proof of Lemma 2.6 with “Lemma 2.2(ii)”, all statements can be verified. ■

## 2.4 General properties of $f$ -sub-Cuntz states

**Lemma 2.8** *Assume that  $A$  is a unital  $C^*$ -algebra and  $B$  is a unital  $C^*$ -subalgebra of  $A$ . For two states  $\omega$  and  $\omega'$  on  $A$ , if  $\omega$  is pure and the restriction  $\pi_\omega|_B$  is irreducible, then  $\omega \sim \omega'$  implies  $\omega|_B \sim \omega'|_B$ .*

*Proof.* Assume  $\omega \sim \omega'$ . Since  $\omega$  is pure,  $\omega'$  is also pure. Hence there exists an irreducible representation  $\pi$  of  $A$  on a Hilbert space  $\mathcal{H}$  with two cyclic unit vectors  $\Omega$  and  $\Omega'$  such that  $\omega = \langle \Omega | \pi(\cdot) \Omega \rangle$  and  $\omega' = \langle \Omega' | \pi(\cdot) \Omega' \rangle$ . By assumption,  $\pi(B)\overline{v} = \mathcal{H}$  for any nonzero vector  $v \in \mathcal{H}$ . Hence  $\pi(B)\Omega = \mathcal{H} = \pi(B)\Omega'$ . From this,  $(\pi(B)\Omega, \pi|_B)$  and  $(\pi(B)\Omega', \pi|_B)$  are unitarily equivalent as representations of  $B$ . Since  $\omega|_B = \langle \Omega | \pi|_B(\cdot) \Omega \rangle$  and  $\omega'|_B = \langle \Omega' | \pi|_B(\cdot) \Omega' \rangle$ , we obtain  $\omega|_B \sim \omega'|_B$ .  $\blacksquare$

Lemma 2.8 does not hold when  $\pi_\omega|_B$  is not irreducible. For example, let  $A := M_3(\mathbb{C}) \curvearrowright \mathbb{C}^3$  and  $B := \mathbb{C} \oplus M_2(\mathbb{C}) \subset A$ , and let  $e_1, e_2, e_3$  denote the standard basis of  $\mathbb{C}^3$ . Define two states  $\omega := \langle e_1 | (\cdot) e_1 \rangle$  and  $\omega' := \langle e_2 | (\cdot) e_2 \rangle$  on  $A$ . Then  $\omega \sim \omega'$ , but  $\omega|_B \not\sim \omega'|_B$ .

Let  $V^{(m)} := \mathbb{C}^m$  when  $m < \infty$  and  $V^{(\infty)} = \ell^2$ , and define  $V_1^{(m)} := \{v \in V^{(m)} : \|v\| = 1\}$  for  $2 \leq m \leq \infty$ . Let  $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$  denote the GNS representation by a state  $\omega$ .

**Corollary 2.9** *For  $2 \leq m \leq \infty$ , fix  $f \in \text{Hom}(\mathcal{O}_m, \mathcal{O}_n)$  and identify  $\mathcal{O}_m$  with  $f(\mathcal{O}_m)$ . For  $z, y \in V_1^{(m)}$ , let  $\omega$  and  $\omega'$  be  $f$ -sub-Cuntz states by  $z$  and  $y$ , respectively such that  $\pi_\omega|_{\mathcal{O}_m}$  is irreducible. If  $\omega \sim \omega'$ , then  $z = y$ .*

*Proof.* Assume  $\omega \sim \omega'$ . Remark that restrictions  $\omega|_{\mathcal{O}_m}$  and  $\omega'|_{\mathcal{O}_m}$  are Cuntz states  $\omega_z$  and  $\omega_y$  on  $\mathcal{O}_m$ , respectively by definition. From this and Lemma 2.8,  $\omega_z = \omega|_{\mathcal{O}_m} \sim \omega'|_{\mathcal{O}_m} = \omega_y$ . This is equivalent to  $z = y$ .  $\blacksquare$

For given two embeddings  $f$  and  $g$ , we show a sufficient condition of the equivalence between  $f$ -sub-Cuntz states and  $g$ -sub-Cuntz states.

**Theorem 2.10** *For  $2 \leq m, l \leq \infty$ , let  $f \in \text{Hom}(\mathcal{O}_m, \mathcal{O}_n)$  and  $g \in \text{Hom}(\mathcal{O}_l, \mathcal{O}_n)$  and let  $t_1, \dots, t_m$  and  $u_1, \dots, u_l$  denote Cuntz generators of  $\mathcal{O}_m$  and  $\mathcal{O}_l$ , respectively. For  $z \in V_1^{(m)}$  and  $y \in V_1^{(l)}$ , assume*

$$f(t(z)) = g(u(y)) \quad (2.5)$$

*for  $t(z) := \sum_{j=1}^m z_j t_j \in \mathcal{O}_m$  and  $u(y) := \sum_{i=1}^l y_i u_i \in \mathcal{O}_l$ .*

- (i) For a state  $\omega$  on  $\mathcal{O}_n$ ,  $\omega$  is an  $f$ -sub-Cuntz state by  $z$  if and only if  $\omega$  is a  $g$ -sub-Cuntz state by  $y$ .
- (ii) If an  $f$ -sub-Cuntz state by  $z$  or a  $g$ -sub-Cuntz state by  $y$  is unique, then they coincide as a state on  $\mathcal{O}_n$ . Especially, they are equivalent.

*Proof.* (i) From Lemma 1.2(ii)((a) $\Leftrightarrow$ (b)) and (2.5), the statement holds.  
(ii) From (i), if one is unique, then so is other. Hence the statement holds from (i). ■

### 3 Proofs of main theorems

#### 3.1 Proofs of Theorem 1.5 and Theorem 1.6

In order to prove Theorem 1.5, we show formulas of explicit values of geometric progression states. Let  $\mathcal{I}_n$  be as in (1.16).

**Theorem 3.1** Fix  $k \geq 2$  and let  $m = (n-1)k + 1$ . For  $z = (z_1, \dots, z_m) \in (\mathbb{C}^m)_1$ , let  $\omega$  be a geometric progression state on  $\mathcal{O}_n$  by  $z$ . If  $|z_m| < 1$ , then the following holds:

- (i) For  $f$  in (1.4), we write  $f(t_i)$  as  $t_i$  for short. For  $J, K \in \mathcal{I}_n$ , there exist unique  $(\hat{J}, a)$  and  $(\hat{K}, b)$  in  $\mathcal{I}_m \times \{0, 1, \dots, k-1\}$  such that  $s_J = t_{\hat{J}} s_n^a$ ,  $s_K = t_{\hat{K}} s_n^b$  and  $\omega(s_J s_K^*) = \overline{z_{\hat{J}}} z_{\hat{K}} \omega(s_n^a (s_n^b)^*)$ .
- (ii) When  $0 \leq a \leq b \leq k-1$ ,

$$\omega(s_n^a (s_n^b)^*) = \sum_{j=1}^{(n-1)(k-b)} \overline{z_{(n-1)a+j}} z_{(n-1)b+j} + \frac{|z_m|^2 Z_{b-a} + \overline{z_m} Z_{k-b+a}}{1 - |z_m|^2} \quad (3.1)$$

where  $Z_c \in \mathbb{C}$  ( $0 \leq c \leq k$ ) is defined as

$$Z_c := \sum_{r=1}^{(n-1)(k-c)} \overline{z_{(n-1)c+r}} z_r \quad (0 \leq c \leq k-1), \quad Z_k := 0. \quad (3.2)$$

*Proof.* (i) From Lemma 2.2(i) and Lemma 1.2(ii)((a) $\Rightarrow$ (c)), the statement holds.

(ii) Let  $\Theta_{a,b} := \omega(s_n^a(s_n^b)^*)$ . Assume  $k-2 \geq b \geq a \geq 0$ . By Corollary 2.1(i)((a) $\Rightarrow$ (c)) and  $\sum_{i=1}^n s_i s_i^* = I$ , we obtain

$$\Theta_{a,b} = \sum_{i=1}^n \omega(s_n^a s_i s_i^* (s_n^b)^*) = \sum_{i=1}^{n-1} \overline{z_{(n-1)a+i}} z_{(n-1)b+i} + \Theta_{a+1,b+1}. \quad (3.3)$$

By repetition to  $\Theta_{a+1,b+1}$ , we obtain

$$\begin{aligned} \Theta_{a,b} &= \sum_{r=a}^{k-1-b+a} \sum_{i=1}^{n-1} \overline{z_{(n-1)r+i}} z_{(n-1)(r+b-a)+i} + \Theta_{k-b+a,k} \\ &= \sum_{j=(n-1)a+1}^{(n-1)(k-b+a)} \overline{z_j} z_{(n-1)(b-a)+j} + z_m \omega(s_n^{k-b+a}). \end{aligned} \quad (3.4)$$

On the other hand, from Corollary 2.1(i)((a) $\Rightarrow$ (b)),

$$\begin{aligned} \omega(s_n^a) &= \sum_{i=1}^{n-1} \sum_{j=a}^{k-1} \overline{z_{(n-1)j+i}} \omega(t_{(n-1)(j-a)+i}^*) + \overline{z_m} \omega((s_n^{k-a})^*) \\ &= Z_a + \overline{z_m \omega(s_n^{k-a})} \end{aligned} \quad (3.5)$$

where we use  $t_{(n-1)j+i}^* s_n^a = t_{(n-1)(j-a)+i}^*$  when  $j \geq a$  and  $t_{(n-1)j+i}^* s_n^a = 0$  when  $j < a$ . By replacing  $a$  with  $k-a$  in (3.5), we obtain  $\omega(s_n^{k-a}) = Z_{k-a} + \overline{z_m \omega(s_n^a)}$ . By substituting this into (3.5), we obtain

$$\omega(s_n^a) = \frac{\overline{z_m Z_{k-a}} + Z_a}{1 - |z_m|^2} \quad (0 \leq a \leq k). \quad (3.6)$$

From this and (3.4), the statement is verified.  $\blacksquare$

**Theorem 3.2** For  $z \in \ell_1^2$ , let  $\omega$  be a geometric progression state on  $\mathcal{O}_n$  by  $z$ . For  $f$  in (1.5), we write  $f(t_i)$  as  $t_i$  for short. For  $J, K \in \mathcal{I}_n$ , there exist unique  $(\hat{J}, a)$  and  $(\hat{K}, b)$  in  $\mathcal{I}_\infty \times \mathbb{Z}_{\geq 0}$  such that  $s_J = t_{\hat{J}} s_n^a$ ,  $s_K = t_{\hat{K}} s_n^b$  and

$$\omega(s_J s_K^*) = \overline{z_{\hat{J}}} z_{\hat{K}} \sum_{j \geq 1} \overline{z_{(n-1)a+j}} z_{(n-1)b+j}. \quad (3.7)$$

*Proof.* Let  $J, K \in \mathcal{I}_n$ . From Lemma 2.2(ii) and Lemma 1.2(ii)((a) $\Rightarrow$ (c)),  $\omega(s_J s_K^*) = \overline{z_J} z_{\hat{K}} \omega(s_n^a (s_n^b)^*)$  for some  $(\hat{J}, a)$  and  $(\hat{K}, b)$ . From Corollary 2.1(ii), we can prove  $\omega(s_n^a (s_n^b)^*) = \sum_{j \geq 1} \overline{z_{(n-1)a+j}} z_{(n-1)b+j}$ . Hence (3.7) is proved. ■

*Proof of Theorem 1.6.* If  $|z_m| = 1$ , then the equation  $z_m \pi(s_n^k) \Omega = \Omega$  holds from Corollary 2.1(i)((a) $\Rightarrow$ (b)). Hence the state is a sub-Cuntz state by  $z_m e_n^{\otimes k} \in (\mathcal{V}_{n,k})_1$  from Lemma 2.5(iii)((b) $\Rightarrow$ (a)). From Theorem 2.4(iii), the statement holds. ■

*Proof of Theorem 1.5.* (i) ( $\Leftarrow$ ) From Theorem 3.1, the statement holds.  
 $(\Rightarrow)$  From Theorem 1.6, the statement holds.  
(ii) From Theorem 3.2, the statement holds. ■

## 3.2 Proof of Theorem 1.8

In order to prove Theorem 1.8, we show the following theorem.

**Theorem 3.3** *For a state  $\omega$ , let  $\pi_\omega$  denote the GNS representation by  $\omega$ .*

- (i) *Identify  $\mathcal{O}_\infty$  with  $f(\mathcal{O}_\infty)$  for  $f$  in (1.5). For  $z \in \ell_1^2$ , let  $\omega'_z$  be as in Theorem 1.5(ii) and let  $\omega_z$  be the Cuntz state on  $\mathcal{O}_\infty$  by  $z$ . Then the restriction  $\pi_{\omega'_z}|_{\mathcal{O}_\infty}$  of  $\pi_{\omega'_z}$  to  $\mathcal{O}_\infty$  is unitarily equivalent to  $\pi_{\omega_z}$ , that is,  $\pi_{\omega'_z}|_{\mathcal{O}_\infty} \sim \pi_{\omega_z}$ . Especially,  $\pi_{\omega'_z}|_{\mathcal{O}_\infty}$  is irreducible.*
- (ii) *Assume  $m = (n-1)k + 1$  for  $k \geq 2$ . Identify  $\mathcal{O}_m$  with  $f(\mathcal{O}_m)$  for  $f$  in (1.4). For  $z \in \mathcal{W}_m$ , let  $\omega'_z$  be as in Theorem 1.5(i) and let  $\omega_z$  be the Cuntz state on  $\mathcal{O}_m$  by  $z$ . Then the restriction  $\pi_{\omega'_z}|_{\mathcal{O}_m}$  of  $\pi_{\omega'_z}$  to  $\mathcal{O}_m$  is unitarily equivalent to  $\pi_{\omega_z}$ , that is,  $\pi_{\omega'_z}|_{\mathcal{O}_m} \sim \pi_{\omega_z}$ . Especially,  $\pi_{\omega'_z}|_{\mathcal{O}_m}$  is irreducible.*

*Proof.* Let  $(\mathcal{H}, \pi, \Omega)$  denote the GNS representation by  $\omega'_z$  and we write  $\pi(s_i)$  as  $s_i$  for short.

(i) It is sufficient to show that  $\mathcal{O}_\infty \Omega$  is dense in  $\mathcal{H}$ . From Lemma 2.7(iii), it suffices to show  $s_J \Omega \in \overline{\mathcal{O}_\infty \Omega}$  for all  $J$ . From Lemma 2.2, for any  $J \in \mathcal{I}_n$ ,  $s_J \Omega = t_J s_n^a \Omega \in \mathcal{O}_\infty s_n^a \Omega$  for some  $(\hat{J}, a)$ . Hence it is enough to show  $s_n^a \Omega \in \overline{\mathcal{O}_\infty \Omega}$  for any  $a$ . Since  $s_n^a = s_n^a \sum_{i=1}^n s_i s_i^* = \sum_{i=1}^{n-1} t_{(n-1)a+i} t_i^* + s_n^{a+1} s_n^*$ , we obtain

$$s_n^a \Omega = \sum_{i=1}^{n-1} \overline{z_i} t_{(n-1)a+i} \Omega + s_n^{a+1} s_n^* \Omega \quad (3.8)$$

where we use Lemma 1.2(ii)((a) $\Rightarrow$ (c)). Since  $s_n^{a+1}s_n^*\Omega = s_n^{a+1}\sum_{i=1}^n s_i s_i^* s_n^*\Omega$  in (3.8), we obtain

$$\begin{aligned}
s_n^a\Omega &= \sum_{i=1}^{n-1} \overline{z(0,i)} t_{(n-1)a+i}\Omega + \sum_{i=1}^{n-1} \overline{z(1,i)} t_{(n-1)(a+1)+i}\Omega + s_n^{a+2}(s_n^2)^*\Omega \\
&= \sum_{r=0}^1 \sum_{i=1}^{n-1} \overline{z(r,i)} t_{(n-1)(a+r)+i}\Omega + s_n^{a+2}(s_n^2)^*\Omega \\
&= \dots \\
&= \sum_{r=0}^{R-1} \sum_{i=1}^{n-1} \overline{z(r,i)} t_{(n-1)(a+r)+i}\Omega + s_n^{a+R}(s_n^R)^*\Omega
\end{aligned} \tag{3.9}$$

for all integer  $R \geq 1$  where  $z(r,i) := z_{(n-1)r+i}$ . Since  $\|s_n^{a+R}(s_n^R)^*\Omega\|^2 = \|(s_n^R)^*\Omega\|^2 = \sum_{r \geq 1} |z_{(n-1)R+r}|^2$  from Theorem 3.2,  $s_n^{a+R}(s_n^R)^*\Omega \rightarrow 0$  when  $R \rightarrow \infty$ . From this and (3.9), we obtain  $s_n^a\Omega = \sum_{r \geq 0} \sum_{i=1}^{n-1} \overline{z(r,i)} t_{(n-1)(a+r)+i}\Omega \in \overline{\mathcal{O}_\infty\Omega}$  for any  $a \geq 1$ .

(ii) From Lemma 2.6(iii), it is sufficient to show  $s_J\Omega \in \mathcal{O}_m\Omega$  for all  $J$ . From Lemma 2.2, for any  $J \in \mathcal{I}_n$ ,  $s_J\Omega = t_{\hat{J}}s_n^a\Omega \in \mathcal{O}_m s_n^a\Omega$  for some  $\hat{J}$  and  $0 \leq a \leq k-1$ . Hence it is sufficient to show  $s_n^a\Omega \in \mathcal{O}_m\Omega$  for any  $0 \leq a \leq k-1$ . By using  $s_n^a = \sum_{i=1}^{n-1} t_{(n-1)a+i}t_i^* + s_n^{a+1}s_n^*$  and the analogy of (3.9), we can prove

$$s_n^a\Omega = \sum_{r=0}^{k-a-1} \sum_{i=1}^{n-1} \overline{z(r,i)} t_{(n-1)(a+r)+i}\Omega + t_m(s_n^{k-a})^*\Omega. \tag{3.10}$$

On the other hand,  $(s_n^{k-a})^*\Omega = \sum_{r=k-a}^{k-1} \sum_{i=1}^{n-1} \overline{z(r,i)} s_n^{r-(k-a)} s_i\Omega + \overline{z_m} s_n^a\Omega$ . By substituting this into (3.10), we obtain

$$s_n^a\Omega = g \sum_{i=1}^{n-1} \left[ \sum_{r=0}^{k-a-1} \overline{z(r,i)} t_{(n-1)(a+r)+i} + t_m \sum_{r=k-a}^{k-1} \overline{z(r,i)} t_{(n-1)(r-(k-a))+i} \right] \Omega \tag{3.11}$$

where  $g := I + \sum_{l \geq 1} (\overline{z_m} t_m)^l \in \mathcal{O}_m$ . Hence  $s_n^a\Omega \in \mathcal{O}_m\Omega$  for all  $0 \leq a \leq k-1$ . ■

*Proof of Theorem 1.8.* (i) It is sufficient to show that  $\omega'_z \sim \omega'_y$  implies  $z = y$ . From Theorem 3.3(ii), this holds from Lemma 2.8 and Corollary 2.9.

(ii) As the same token with the proof of (i), the statement holds from Theorem 3.3(i). ■

### 3.3 Proof of Theorem 1.9

Assume  $m = (n - 1)k + 1$  for  $k \geq 1$ . Let  $s_1, \dots, s_n, u_1, \dots, u_m$  and  $t_1, t_2, \dots$  denote Cuntz generators of  $\mathcal{O}_n, \mathcal{O}_m$  and  $\mathcal{O}_\infty$ , respectively. Identify  $\mathcal{O}_m$  and  $\mathcal{O}_\infty$  with images of geometric progression embeddings in  $\mathcal{O}_n$ .

*Proof of Theorem 1.9(i).* Let  $t(z) := \sum_{i \geq 1} z_i t_i$  and  $s(y) := \sum_{i=1}^n y_i s_i$ .

( $\Rightarrow$ ) Assume  $\omega'_z \sim \omega_y$ . Then we can assume that  $\mathcal{O}_n$  acts on a Hilbert space  $\mathcal{H}$  with two cyclic unit vectors  $\Omega$  and  $\Omega'$  such that  $t(z)\Omega = \Omega$  and  $s(y)\Omega' = \Omega'$ . Remark that  $\omega'_z = \langle \Omega | (\cdot) \Omega \rangle$  and  $\omega_y = \langle \Omega' | (\cdot) \Omega' \rangle$  from Corollary 2.1(ii)((b) $\Rightarrow$ (a)) and Lemma 2.5(iii)((b) $\Rightarrow$ (a)). Let  $X := \langle \Omega | \Omega' \rangle$ . Then

$$X = \langle t(z)\Omega | \Omega' \rangle = \sum_{i \geq 1} \bar{z}_i \langle t_i \Omega | \Omega' \rangle = \alpha X \quad (3.12)$$

where  $\alpha := \sum_{r \geq 0} \sum_{j=1}^{n-1} \overline{z_{(n-1)r+j}} y_n^r y_j \in \mathbb{C}$ . Since  $\langle s_J \Omega | \Omega' \rangle = y_J X$  for all  $J$ , if  $X = 0$ , then  $\omega'_z \not\sim \omega_y$ . Hence  $X \neq 0$ . From this and (3.12),  $\alpha = 1$ . This implies  $|y_n| < 1$ . Hence  $\|\tilde{y}\| = 1$  and we can write  $\alpha = \langle z | \tilde{y} \rangle$ . From  $\langle z | \tilde{y} \rangle = 1$  and  $\|z\| = \|\tilde{y}\| = 1$ , we obtain  $z = \tilde{y}$ .

( $\Leftarrow$ ) Assume that  $|y_n| < 1$  and  $z = \tilde{y}$ . Let  $(\mathcal{H}, \pi, \Omega')$  denote the GNS representation by  $\omega_y$ . Then  $\pi(s(y))\Omega' = \Omega'$ . By assumption, we can verify  $\pi(t(z))\Omega' = \Omega'$ . From this and Corollary 2.1(ii)((b) $\Rightarrow$ (a)),  $\omega'_z = \omega_y$ . Especially,  $\omega'_z \sim \omega_y$ .  $\blacksquare$

*Proof of Theorem 1.9(ii).* ( $\Rightarrow$ ) In  $\mathcal{O}_n$ , we obtain  $t_{(m-1)r+i} = u_m^r u_i$  for  $r \geq 0$  and  $i = 1, \dots, m-1$ . From this, we can verify that  $\omega'_z|_{\mathcal{O}_m}$  is the geometric progression state on  $\mathcal{O}_m$  by  $z$ . From Theorem 3.3(ii) and Lemma 2.8,  $\omega'_z \sim \omega'_y$  implies

$$\omega'_z|_{\mathcal{O}_m} \sim \omega'_y|_{\mathcal{O}_m}. \quad (3.13)$$

By definition,  $\omega'_y|_{\mathcal{O}_m}$  is the Cuntz state on  $\mathcal{O}_m$  by  $y$ . Applying Theorem 1.9(i) to (3.13) by replacing  $(\mathcal{O}_n, \mathcal{O}_\infty)$  with  $(\mathcal{O}_m, \mathcal{O}_\infty)$ , we obtain  $z = \tilde{y}$ .

( $\Leftarrow$ ) Assume that  $\mathcal{O}_n$  acts on a Hilbert space  $\mathcal{H}$  with a cyclic unit vector  $\Omega'$  such that  $u(y)\Omega' = \Omega'$  where  $u(y) := \sum_{j=1}^m y_j u_j$ . Then  $\omega'_y = \langle \Omega' | (\cdot) \Omega' \rangle$ . Define  $Y := y_m$  and  $y' := y - Y e_m$ . If  $z = \tilde{y}$ , then we obtain

$$t(z) = t(\tilde{y}) = \sum_{r \geq 0} Y^r u_m^r u(y') \quad (3.14)$$

where we use  $t_{(m-1)r+j} = u_m^r u_j$ . Let  $U := Y u_m \in \mathcal{O}_m \subset \mathcal{O}_n$ . Then we obtain  $t(z) = (I - U)^{-1} u(y')$ . From (3.14) and  $u(y')\Omega' = (I - Y u_m)\Omega' = (I - U)\Omega'$ , we

obtain  $t(z)\Omega' = (I - U)^{-1}u(y')\Omega' = \Omega'$ . From this, Lemma 1.2(ii)((b) $\Rightarrow$ (a)) and the uniqueness of  $\omega'_z$ , we obtain  $\omega'_z = \omega'_y$ . Especially,  $\omega'_z \sim \omega'_y$ . ■

### 3.4 Proofs of Theorem 1.10 and Theorem 1.11

*Proof of Theorem 1.10.* (i) Since  $l = (m - 1)k + 1 = (n - 1)ak + 1$ , there exists the geometric progression embedding of  $\mathcal{O}_l$  into  $\mathcal{O}_n$ . By assumption and the choice of  $z$ ,  $|\hat{z}_l| = |z_m^k| < 1$ . Hence  $\hat{z} \in \mathcal{W}_l$ . Let  $t_1, \dots, t_m$  and  $u_1, \dots, u_l$  denote Cuntz generators of  $\mathcal{O}_m$  and  $\mathcal{O}_l$ , respectively. Identify  $\mathcal{O}_m$  and  $\mathcal{O}_l$  with images of geometric progression embeddings in  $\mathcal{O}_n$ , respectively. Let  $T := \sum_{r=0}^{k-1} \sum_{i=1}^{m-1} z_m^r z_i t_m^r t_i + z_m^k t_m^k \in \mathcal{O}_m \subset \mathcal{O}_n$ . Then we see  $T = u(\hat{z}) := \sum_{j=1}^l \hat{z}_j u_j \in \mathcal{O}_l \subset \mathcal{O}_n$ . Let  $z' := z - z_m e_m \in \mathbb{C}^m$  and  $Z := z_m t_m \in \mathcal{O}_m \subset \mathcal{O}_n$ . Then we can rewrite  $T = \sum_{r=0}^{k-1} (z_m t_m)^r t(z') + (z_m t_m)^k = (I - Z^k)(I - Z)^{-1}t(z') + Z^k$ . Let  $(\mathcal{H}, \pi, \Omega)$  denote the GNS representation by  $\omega'_z$ . We write  $\pi(s_i)$  as  $s_i$  for short. Since  $\Omega = t(z)\Omega = \{t(z') + z_m t_m\}\Omega$ ,  $t(z')\Omega = (I - z_m t_m)\Omega = (I - Z)\Omega$ . By using these, we obtain  $u(\hat{z})\Omega = T\Omega = \{(I - Z^k)(I - Z)^{-1}t(z') + Z^k\}\Omega = \Omega$ . From this, Lemma 1.2(ii)((b) $\Rightarrow$ (a)) and the uniqueness of  $\omega'_{\hat{z}}$ , the statement holds.

(ii) From (i), the statement holds.

(iii) By assumption,  $m = (n - 1)a + 1$  and  $l = (n - 1)b + 1$  for some  $a, b \geq 1$ . Since  $p = (n - 1)^2 ab + 1$ , we can define the geometric progression embedding of  $\mathcal{O}_p$  into  $\mathcal{O}_n$ . Therefore  $\mathcal{S}_{m,n} \cup \mathcal{S}_{l,n} \subset \mathcal{S}_{p,n}$  from (ii). From (1.13), we obtain  $\hat{z}, \hat{y} \in \mathcal{W}_p$  such that  $\omega'_z = \omega'_{\hat{z}}$  and  $\omega'_y = \omega'_{\hat{y}}$ . Hence  $\omega'_z \sim \omega'_y$  if and only if  $\omega'_{\hat{z}} \sim \omega'_{\hat{y}}$ . This is equivalent to  $\hat{z} = \hat{y}$  from Theorem 1.8(i). Since  $\hat{z}$  and  $\hat{y}$  coincide with the l.h.s. and the r.h.s. of (1.14), respectively, the statement holds.

(iv) From Theorem 1.9(ii),  $\omega'_z \sim \omega'_{\tilde{z}}$ . Hence  $\omega'_z \sim \omega_y$  is equivalent to  $\omega'_{\tilde{z}} \sim \omega_y$ . From Theorem 1.9(i), this is equivalent that  $|y_n| < 1$  and  $\tilde{z} = \tilde{y}$ . By definitions of  $\tilde{z}$ ,  $\tilde{y}$  and  $\hat{y}$ , we can verify that this is equivalent that  $|y_n| < 1$  and  $z = \hat{y}$ . ■

*Proof of Theorem 1.11.* From Lemma 2.6(i), the statement holds. ■



## 4 Examples

### 4.1 Geometric progression states on $\mathcal{O}_2$

In this subsection, we show definitions and theorems for geometric progression states on  $\mathcal{O}_2$  of order 2 and  $\infty$  as examples of main theorems. Let  $s_1, s_2$  denote Cuntz generators of  $\mathcal{O}_2$ .

#### 4.1.1 Case of order 2

We summarize properties of geometric progression states on  $\mathcal{O}_2$  of order 2. Define  $(\mathbb{C}^3)_1 := \{(w_1, w_2, w_3) \in \mathbb{C}^3 : |w_1|^2 + |w_2|^2 + |w_3|^2 = 1\}$ . By definition,  $\omega$  is a geometric progression state on  $\mathcal{O}_2$  by  $z = (z_1, z_2, z_3) \in (\mathbb{C}^3)_1$  if and only if  $\omega$  satisfies the following equations:

$$\omega(s_1) = \overline{z_1}, \quad \omega(s_2 s_1) = \overline{z_2}, \quad \omega(s_2^2) = \overline{z_3}. \quad (4.1)$$

For a state  $\omega$  on  $\mathcal{O}_2$  with the GNS representation  $(\mathcal{H}, \pi, \omega)$ , the following are equivalent from Corollary 2.1(i):

- (i)  $\omega$  is a geometric progression state by  $z$ .
- (ii)  $\{z_1 \pi(s_1) + z_2 \pi(s_2 s_1) + z_3 \pi(s_2^2)\} \Omega = \Omega$ .
- (iii)  $\pi(s_1)^* \Omega = z_1 \Omega$ ,  $\pi(s_2 s_1)^* \Omega = z_2 \Omega$  and  $\pi(s_2^2)^* \Omega = z_3 \Omega$ .

**Corollary 4.1** (i) *A geometric progression state on  $\mathcal{O}_2$  by  $z = (z_1, z_2, z_3) \in (\mathbb{C}^3)_1$  is unique if and only if  $|z_3| < 1$ . In this case, it is pure and we write such a state as  $\omega'_z$ .*

- (ii) *For  $z, y \in \mathcal{W}_3 := \{(w_1, w_2, w_3) \in (\mathbb{C}^3)_1 : |w_3| < 1\}$ ,  $\omega'_z \sim \omega'_y$  if and only if  $z = y$ .*
- (iii) *For  $z \in \mathcal{W}_3$ ,  $\omega'_z$  is equivalent to the Cuntz state  $\omega_y$  on  $\mathcal{O}_2$  by  $y = (y_1, y_2) \in (\mathbb{C}^2)_1$  if and only if  $|y_2| < 1$  and  $z = (y_1, y_2 y_1, y_2^2)$ .*

*Proof.* (i) See Theorem 1.5(i). (ii) See Theorem 1.8(i). (iii) See Theorem 1.10(iv). ■

An idea of this study was brought from the following example. Abe [1] constructed a representation  $(\mathcal{H}, \pi)$  of  $\mathcal{O}_2$  with a cyclic vector  $\Omega \in \mathcal{H}$  which satisfies

$$\frac{1}{\sqrt{2}}\pi(s_1 + s_2s_1)\Omega = \Omega. \quad (4.2)$$

By generalizing this, we obtained a class of representations of  $\mathcal{O}_n$  and showed their properties ([30], Lemma 2.1). From Corollary 4.1, the representation in (4.2) is unitarily equivalent to the GNS representation by the geometric progression state  $\omega'_z$  on  $\mathcal{O}_2$  for  $z = (1/\sqrt{2}, 1/\sqrt{2}, 0) \in \mathbb{C}^3$  and it is irreducible.

#### 4.1.2 Case of order $\infty$

A state  $\omega$  on  $\mathcal{O}_2$  is a geometric progression state by  $z = (z_1, z_2, \dots) \in \ell_1^2 := \{(z_i) : z_i \in \mathbb{C} \text{ for all } i \text{ and } \sum_{i \geq 1} |z_i|^2 = 1\}$  if and only if  $\omega$  satisfies  $\omega(s_2^{n-1}s_1) = \overline{z_n}$  for all  $n \geq 1$ . From Corollary 2.1(ii), this is equivalent that  $(z_1s_1 + z_2s_2s_1 + z_3s_2^2s_1 + \dots)\Omega = \Omega$ , or  $\pi(s_2^{n-1}s_1)^*\Omega = z_n\Omega$  for all  $n \geq 1$  where  $(\mathcal{H}, \pi, \Omega)$  denotes the GNS representation by  $\omega$ .

**Corollary 4.2** (i) For any  $z \in \ell_1^2$ , a geometric progression state on  $\mathcal{O}_2$  by  $z$  exists uniquely and is pure. We write it as  $\omega'_z$ .

(ii) For  $z, y \in \ell_1^2$ ,  $\omega'_z \sim \omega'_y$  if and only if  $z = y$ .

(iii) For  $y = (y_1, y_2) \in (\mathbb{C}^2)_1$ , let  $\omega_y$  denote the Cuntz state on  $\mathcal{O}_2$  by  $y$ . For  $z \in \ell_1^2$ ,  $\omega'_z \sim \omega_y$  if and only if  $|y_2| < 1$  and  $z = (y_1, y_2y_1, y_2^2y_1, y_2^3y_1, \dots)$ . In this case,  $\omega'_z = \omega_y$ .

(iv) For  $z \in \ell_1^2$  and  $y = (y_1, y_2, y_3) \in \mathcal{W}_3$ ,  $\omega'_z \sim \omega'_y$  if and only if

$$z = (y_1, y_2, y_3y_1, y_3y_2, y_3^2y_1, y_3^2y_2, \dots). \quad (4.3)$$

In this case,  $\omega'_z = \omega'_y$ .

*Proof.* (i) See Theorem 1.5(ii). (ii) See Theorem 1.8(ii). (iii) See Theorem 1.9(i). (iv) See Theorem 1.9(ii). ■

From Corollary 4.2(iii),  $\omega'_z \sim \omega_y$  if and only if the sequence  $z$  is just a geometric progression of complex numbers with initial value  $y_1$  and common ratio  $y_2$  such that  $|y_1|^2 + |y_2|^2 = 1$  and  $|y_2| < 1$ . In this sense, we can regard geometric progression states as generalizations of geometric progressions of complex numbers.

## 4.2 Typical examples

**Example 4.3** Let  $U(1) := \{c \in \mathbb{C} : |c| = 1\}$ . As examples of Corollary 4.1, we show three kinds of states on  $\mathcal{O}_2$ . For  $c \in U(1)$ , three states  $\rho_c, \rho'_c, \rho''_c$  on  $\mathcal{O}_2$  which satisfy

$$\rho_c(s_1) = c, \quad \rho'_c(s_2 s_1) = c, \quad \rho''_c(s_1 + s_2 s_1) = \sqrt{2}c \quad (4.4)$$

exist uniquely and are pure. They are the Cuntz state, the sub-Cuntz state and the geometric progression state on  $\mathcal{O}_2$  by  $(\bar{c}, 0)$ ,  $(0, \bar{c}) \otimes (1, 0)$  and  $(\bar{c}2^{-1/2}, \bar{c}2^{-1/2}, 0)$ , respectively. Any two distinct states in the set  $\{\rho_c, \rho'_c, \rho''_c : c \in U(1)\}$  are not equivalent. They can be unified as the geometric progression state  $\rho_{c_1, c_2}$  which satisfies

$$\rho_{c_1, c_2}(c_1 s_1 + c_2 s_2 s_1) = 1 \quad ((c_1, c_2) \in (\mathbb{C}^2)_1). \quad (4.5)$$

In fact, we see that  $\rho_c = \rho_{\bar{c}, 0}$ ,  $\rho'_c = \rho_{0, \bar{c}}$  and  $\rho''_c = \rho_{\bar{c}2^{-1/2}, \bar{c}2^{-1/2}}$  for  $c \in U(1)$ . Any two distinct states in the set  $\{\rho_{c_1, c_2} : (c_1, c_2) \in (\mathbb{C}^2)_1\}$  are not equivalent. From this, general geometric progression states can be regarded as interpolations between two (or many) special sub-Cuntz states of different orders.

Next, we show an example of Corollary 4.2. Let  $\zeta(x)$  denote the Riemann zeta function for a positive real number  $x > 1$ , that is,  $\zeta(x) := \sum_{n=1}^{\infty} n^{-x}$ .

**Proposition 4.4** Fix  $x > 1$ . Assume that a state  $\omega$  on  $\mathcal{O}_2$  satisfies

$$\sum_{n=1}^{\infty} \frac{\omega(s_2^{n-1} s_1)}{n^{x/2}} = \sqrt{\zeta(x)}. \quad (4.6)$$

Then the following holds.

- (i) For any  $x$ ,  $\omega$  exists uniquely and is pure. We write  $\omega$  as  $\kappa_x$ . Then  $\kappa_x \sim \kappa_{x'}$  if and only if  $x = x'$ .
- (ii) Let  $\rho_{c_1, c_2}$  be as in Example 4.3. For any  $x$  and  $(c_1, c_2) \in (\mathbb{C}^2)_1$ ,  $\kappa_x \not\sim \rho_{c_1, c_2}$ .

*Proof.* (i) Define  $z(x) := (z_n(x)) \in \ell^2$  by  $z_n(x) := \{\zeta(x)n^x\}^{-1/2} \in \mathbb{R}$  for  $n \geq 1$ . Then we see  $\|z(x)\|^2 = 1$ . Hence  $z(x) \in \ell_1^2$ . From this and the definition,  $\omega$  is the geometric progression state on  $\mathcal{O}_2$  by  $z(x)$ . From Corollary 4.2(i), the uniqueness and purity hold. From Corollary 4.2(ii), the equivalence condition holds because  $z(x) = z(x')$  if and only if  $x = x'$ .

(ii) By definition, we see that  $\rho_{c_1, c_2}$  is the geometric progression state on  $\mathcal{O}_2$  by  $(c_1, c_2, 0) \in \mathcal{W}_3$ . From Corollary 4.2(iv), the statement holds.  $\blacksquare$

### 4.3 Transformations of geometric progression states

We show transformations of geometric progression states of order 2 and their equivalence. Let  $f$  be as in (1.4) for  $m = 2n - 1$ . Let  $s_1, \dots, s_n$  and  $t_1, \dots, t_{2n-1}$  denote Cuntz generators of  $\mathcal{O}_n$  and  $\mathcal{O}_{2n-1}$ , respectively. Define  $\alpha \in \text{Aut}\mathcal{O}_n$  and  $\beta \in \text{Aut}\mathcal{O}_{2n-1}$  by

$$\alpha(s_i) := s_{n-i+1} \quad (i = 1, \dots, n), \quad \beta(t_j) := t_{2n-j} \quad (j = 1, \dots, 2n-1). \quad (4.7)$$

Then  $\alpha^{-1} = \alpha$  and  $\beta^{-1} = \beta$ . Define  $f' \in \text{Hom}(\mathcal{O}_{2n-1}, \mathcal{O}_n)$  by  $f' := \alpha \circ f \circ \beta$ . Then we can verify

$$(f'(t_1), \dots, f'(t_{2n-1})) = (s_1^2, s_1 s_2, s_1 s_3, \dots, s_1 s_n, s_2, s_3, \dots, s_n). \quad (4.8)$$

By definition,  $\omega$  is an  $f'$ -sub-Cuntz state on  $\mathcal{O}_n$  by  $z = (z_1, \dots, z_{2n-1}) \in (\mathbb{C}^{2n-1})_1$  if and only if  $\omega$  is a state on  $\mathcal{O}_n$  which satisfies

$$\begin{cases} \omega(s_1^2) = \overline{z_1}, \omega(s_1 s_2) = \overline{z_2}, \dots, \omega(s_1 s_n) = \overline{z_n}, \\ \omega(s_2) = \overline{z_{n+1}}, \dots, \omega(s_n) = \overline{z_{2n-1}}. \end{cases} \quad (4.9)$$

**Proposition 4.5** *Let  $z = (z_1, \dots, z_{2n-1}) \in (\mathbb{C}^{2n-1})_1$ .*

(i) *For an  $f'$ -sub-Cuntz state  $\omega$  on  $\mathcal{O}_n$  by  $z$ , the following holds:*

(a)  *$\omega \circ \alpha$  is the geometric progression state by  $(z_{2n-1}, z_{2n-2}, \dots, z_1)$ .*

(b)  *$\omega$  is unique if and only if  $|z_1| < 1$ . In this case,  $\omega$  is pure and we write  $\omega$  as  $\eta_z$ .*

(ii) *Define  $\mathcal{W}'_{2n-1} := \{(w_1, \dots, w_{2n-1}) \in (\mathbb{C}^{2n-1})_1 : |w_1| < 1\}$ . For  $z, y \in \mathcal{W}'_{2n-1}$ ,  $\eta_z \sim \eta_y$  if and only if  $z = y$ .*

*Proof.* (i) (a) By assumption, the statement holds.  
 (b) From (a) and Theorem 1.5(i), the statement holds.  
 (ii) From (i)(a) and Theorem 1.8(i), the statement holds. ■

## Appendix

### A Proof of Lemma 2.2

In order to prove Lemma 2.2, we show a lemma associated with a free semi-group and its subsemigroups. Fix  $2 \leq n < \infty$ . Let  $s_1, \dots, s_n$  denote Cuntz generators of  $\mathcal{O}_n$ . Let  $\mathcal{I}_n$  be as in (1.16) and  $\mathbf{S}_n := \{s_J : J \in \mathcal{I}_n, J \neq \emptyset\}$ . Then  $\mathbf{S}_n$  is the non-selfadjoint subsemigroup of  $\mathcal{O}_n$  generated by  $s_1, \dots, s_n$  without unit, and it is free [23, 45]. For two nonempty subsets  $X, Y \subset \mathbf{S}_n$ , define  $XY := \{xy : x \in X, y \in Y\}$  and  $X^0 := \{I\}$ ,  $X^a := X^{a-1}X$  for  $a \geq 1$ . Let  $A$  and  $B$  denote subsemigroups of  $\mathbf{S}_n$  generated by  $\{s_1, \dots, s_{n-1}\}$  and  $\{s_n\}$ , respectively:

$$A := \langle \{s_1, \dots, s_{n-1}\} \rangle, \quad B := \langle \{s_n\} \rangle = \{s_n^a : a \geq 1\}. \quad (\text{A.1})$$

Since  $\mathbf{S}_n$  coincides with the free product  $A * B$  [23, 45], any  $x \in \mathbf{S}_n$  belongs to one of the following:

$$A(BA)^a, \quad A(BA)^a B, \quad (BA)^{a'}, \quad (BA)^a B \quad (\text{A.2})$$

for some  $a \geq 0$  and  $a' \geq 1$ .

By identifying  $t_i$  with  $f(t_i)$  in Definition 1.4, we regard  $t_1, t_2, \dots \in \mathcal{O}_n$ . Let  $\mathbf{T}_m$  denote the subsemigroup of  $\mathbf{S}_n$  generated by  $t_1, t_2, \dots$ . Then  $\mathbf{T}_m$  is also free. Since  $t_i = s_i$  for  $i = 1, \dots, n-1$ ,  $A \subset \mathbf{T}_m$ .

**Lemma A.1** *For any  $2 \leq m \leq \infty$  and  $a \geq 1$ ,  $(BA)^a \subset \mathbf{T}_m$ .*

*Proof.* It is sufficient to show  $BA \subset \mathbf{T}_m$ .

(i) Assume  $m < \infty$ . If  $x \in BA$ , then  $x = s_n^a y$  for some  $a \geq 1$  and  $y \in A$ . Then we can write  $a = a'k + a''$  and  $y = s_i y'$  for some  $a' \geq 0$ ,  $0 \leq a'' \leq k-1$ ,  $(a', a'') \neq (0, 0)$ , and some  $1 \leq i \leq n-1$  and  $y' \in A \cup \{I\}$ . From these and  $A \subset \mathbf{T}_m$ ,  $x = s_n^{a'k+a''}(s_i y') = t_m^{a'} t_{(n-1)a''+i} y' \in \mathbf{T}_m$ .  
 (ii) Assume  $m = \infty$ . If  $x \in BA$ , then  $x = s_n^a y$  for some  $a \geq 1$  and  $y \in A$ . Then we can write  $y = s_i y'$  for some  $1 \leq i \leq n-1$  and  $y' \in A \cup \{I\}$ . From

these and  $A \subset \mathbb{T}_\infty$ ,  $x = s_n^a s_i y' = t_{(n-1)a+i} y' \in \mathbb{T}_\infty$ . ■

*Proof of Lemma 2.2.* (i) Assume  $m < \infty$ . Let  $\mathbb{T} := \mathbb{T}_m$  and  $x = s_J$ . If  $x \in \mathbb{T}$ , then the statement holds because  $\mathbb{T} = \mathbb{T}_n^0$ . We check the statement for each case in (A.2) as follows.

- (a) Assume  $x \in A(BA)^a$  for some  $a \geq 0$ . Since  $A \subset \mathbb{T}$ ,  $x \in A(BA)^a \subset \mathbb{T}$  from Lemma A.1. Hence the statement holds.
- (b) Assume  $x \in A(BA)^a B$  for some  $a \geq 0$ . From  $A \subset \mathbb{T}$  and Lemma A.1, we can write  $x = x' s_n^b$  for some  $x' \in \mathbb{T}$  and  $b \geq 1$ . Then we can write  $b = b'k + b''$  for some  $b' \geq 0$ ,  $0 \leq b'' \leq k-1$  and  $(b', b'') \neq (0, 0)$ . Then  $x = x' s_n^{b'k+b''} = x' t_m^{b'} s_n^{b''} \in \mathbb{T} s_n^{b''}$ . Hence the statement holds.
- (c) Assume  $x \in (BA)^{a'}$  for some  $a' \geq 1$ . From Lemma A.1, the statement holds.
- (d) Assume  $x \in (BA)^a B$  for some  $a \geq 0$ . Along with (b), the statement holds.

(ii) Let  $\mathbb{T} := \mathbb{T}_\infty$ . Remark that if  $x \in \mathbb{T}$ , then the statement holds because  $\mathbb{T} = \mathbb{T}_n^0$ . Except the case of (b) in the proof of (i), the statement for each case in (A.2) holds as (i). Assume  $x \in A(BA)^a B$  for some  $a \geq 1$ . From  $A \subset \mathbb{T}$  and Lemma A.1, we can write  $x = x' s_n^b$  for some  $x' \in \mathbb{T}$  and  $b \geq 1$ . Hence the statement holds. ■

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